mobile communications series

SOFTWARE-DEFINED RADIO for ENGINEERS

TRAVIS F. COLLINS
ROBIN GETZ
DI PU
ALEXANDER M. WYGLINSKI

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Software-Defined Radio for Engineers

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Dedication

To my wife Lauren
—Travis Collins

To my wonderful children, Matthew, Lauren, and Isaac, and my patient wife, Michelle—sorry I have been hiding in the basement working on this book. To all my fantastic colleagues at Analog Devices: Dave, Michael, Lars-Peter, Andrei, Mihai, Travis, Wyatt and many more, without whom Pluto SDR and IIO would not exist.
—Robin Getz

To my lovely son Aidi, my husband Di, and my parents Lingzhen and Xuexun
—Di Pu

To my wife Jen
—Alexander Wyglinski

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CHAPTER 3

Probability in Communications

The previous chapter provided us with a range of insights, tools, and algorithms for the modeling and processing of signals and systems from a purely deterministic perspective. However, there are numerous elements within the communication system environment that possess random characteristics, such as the digital values of the binary data stream from an information source, the noise introduced to a transmission when transversing a communication channel, and the sources of interference resulting from multiple users operating within the same environment. Whereas Chapter 2 presented a collection of useful tools for the understanding and modeling of deterministic signals and systems, in this chapter we will introduce a range of techniques and approaches that can be used to model and study probabilistic signals and systems. Starting with an introduction to the concept of both the continuous and *discrete random variable*, this chapter will proceed with an explanation of time-varying random phenomena, called *random processes*, followed by the modeling of various random noise channels.

3.1 Modeling Discrete Random Events in Communication Systems

A discrete random variable represents some sort of behavior occuring within the communication system where the outcome is not absolutely known. For instance, the next value produced by a binary information source has the possibility of producing one of two outputs: a binary 1 or a binary 0. Although we know that either value is possible, we do not have definite knowledge that one of these values will be specifically be produced at the output of the information source. Consequently, we model the output values to be produced by the binary information source as a random variable. The reason we call this random variable a discrete random variable is that it produces a single possible output value from a finite number of options.

To mathematically model this discrete random varaible, suppose we define X such that there exists a distinct set of real numbers x_i that it can produce, each with a specific probability of that value occuring:

$$\sum_i P(X = x_i) = 1, \tag{3.1}$$

where $P(X = x_i)$ is the probability that the random variable will produce an output value x_i. Using the law of total probability [1], we know that

$$P(X \in B) = \sum_{i:x_i \in B} P(X = x_i), \tag{3.2}$$

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where the set B is composed of a collection of values x_i. A specific form of discrete random variable is the *integer-valued random variable*, where its output values are the integers $x_i = i$; that is,

$$P(X \in B) = \sum_{i \in B} P(X = i). \tag{3.3}$$

Since each output of the random variable X possesses a different probability of occurrence, we usually define the probability of a specific discrete output x_i being generated by X as

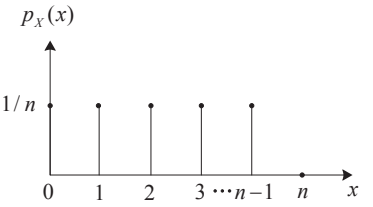
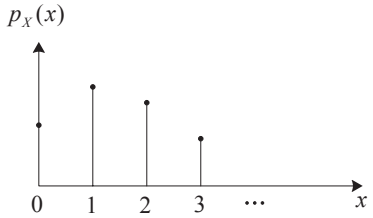
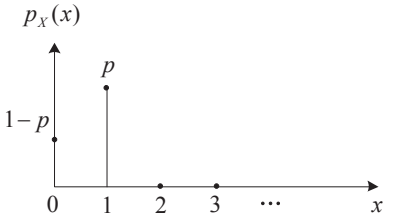
$$p_X(x_i) = P(X = x_i), \tag{3.4}$$

where $p_X(x_i)$ is referred to as the *probability mass function* (PMF). Note that the values of the PMF are constrained by

$$0 \le p_X(x_i) \le 1 \text{ and } \sum_i p_X(x_i) = 1. \tag{3.5}$$

Several frequently used PMFs are specified in Table 3.1, including uniform, Poisson, and Bernoulli random variables. In particular, Bernoulli random variables are used to generate random outputs for binary information sources, while Poisson random variables are often used to model the delays of routing packets in computer networks.

Table 3.1 Several Frequently Used Probability Mass Functions

Random Variable	*PMF Definition*	*Graphical Representation*
Uniform	$p_X(k) = \begin{cases} \frac{1}{n}, & k = 1, \ldots, n \\ 0, & \text{otherwise} \end{cases}$	
Poisson	$p_X(k) = \frac{\lambda^k e^{-\lambda}}{k!}$, $k = 0, 1, 2, \ldots$	
Bernoulli	$p_X(k) = \begin{cases} p, & k = 1 \\ 1-p, & k = 0 \\ 0, & \text{otherwise} \end{cases}$	

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Supposed we now explore how to use these discrete random variables to model an actual component of a communication system; namely, a binary information source. One key characteristic that needs to be incorporated into this tool is the percentage of ones and zeros produced by the *random number generator*, otherwise referred to as an RNG. In the MATLAB script in Code 3.1, a binary RNG is implemented where three vectors are produced. Vector `b1` possesses an equal balance of one and zero values, while vectors `b2` and `b3` generate ratios of 60/40 and 20/80 in terms of ones and zeros, respectively. Note that the MATLAB `rand` uniform RNG function is used to produce the random values between zero and one, which is then rounded to the nearest integer; namely, one or zero.

Code 3.1 Information Source: **`random_example.m`**

```
% Create three binary data streams of length L, each with a different
% percentage of 1 and 0 values
L = 100;
prob1 = 0.5; prob2 = 0.6; prob3 = 0.2; %Probability values for 1 outputs
b1 = round(0.5*rand(1,L)+0.5*prob1); %Even split between 1 and 0 values
b2 = round(0.5*rand(1,L)+0.5*prob2); %Have 60% 1 values and 40% 0 values
b3 = round(0.5*rand(1,L)+0.5*prob3); %Have 20% 1 values and 80% 0 values
```

Manipulating how the output values of the `rand` get rounded to either zero or one by biasing all the values to be closer to one or the other, we can generate binary random values with different percentages of ones and zeros, as shown in Figure 3.1. Using the `stem` command, we can visualize the actual binary values shown in Figure 3.1(a), Figure 3.1(c), and Figure 3.1(e). Although this gives us a general observation about the behavior of these binary RNGs, it is very difficult to distinguish the actual percentages of ones and zeros within the vector. Hence, using a histogram and a very long sequence of randomly generated values, we can characterize these percentages, as shown in Figure 3.1(b), Figure 3.1(d), and Figure 3.1(f). Notice how the histograms accurately show the percentages of ones and zeros in an outputted vector. One important caveat: Since our characterization is dependent on the observation of a random phenomenon, you will need to observe a very substantial amount of data in order to accurately characterize it.

3.1.1 Expectation

Since we are unable to exactly know the output values of these random phenomena that form part of the communication system and its environment, we must instead opt for mathematicallly characterizing the statistical behavior of these random variables. One way to characterize a random variable is by its expected value or mean, which can be quantitatively determined using the expression

$$m_X = E[X] = \sum_i x_i P(X = x_i), \tag{3.6}$$

where the sum of each value is weighted by its probability of occurrence. Consequently, using the definition of the PMF, we can rewrite (3.6) as

$$E[X] = \sum_i x_i p_X(x_i). \tag{3.7}$$

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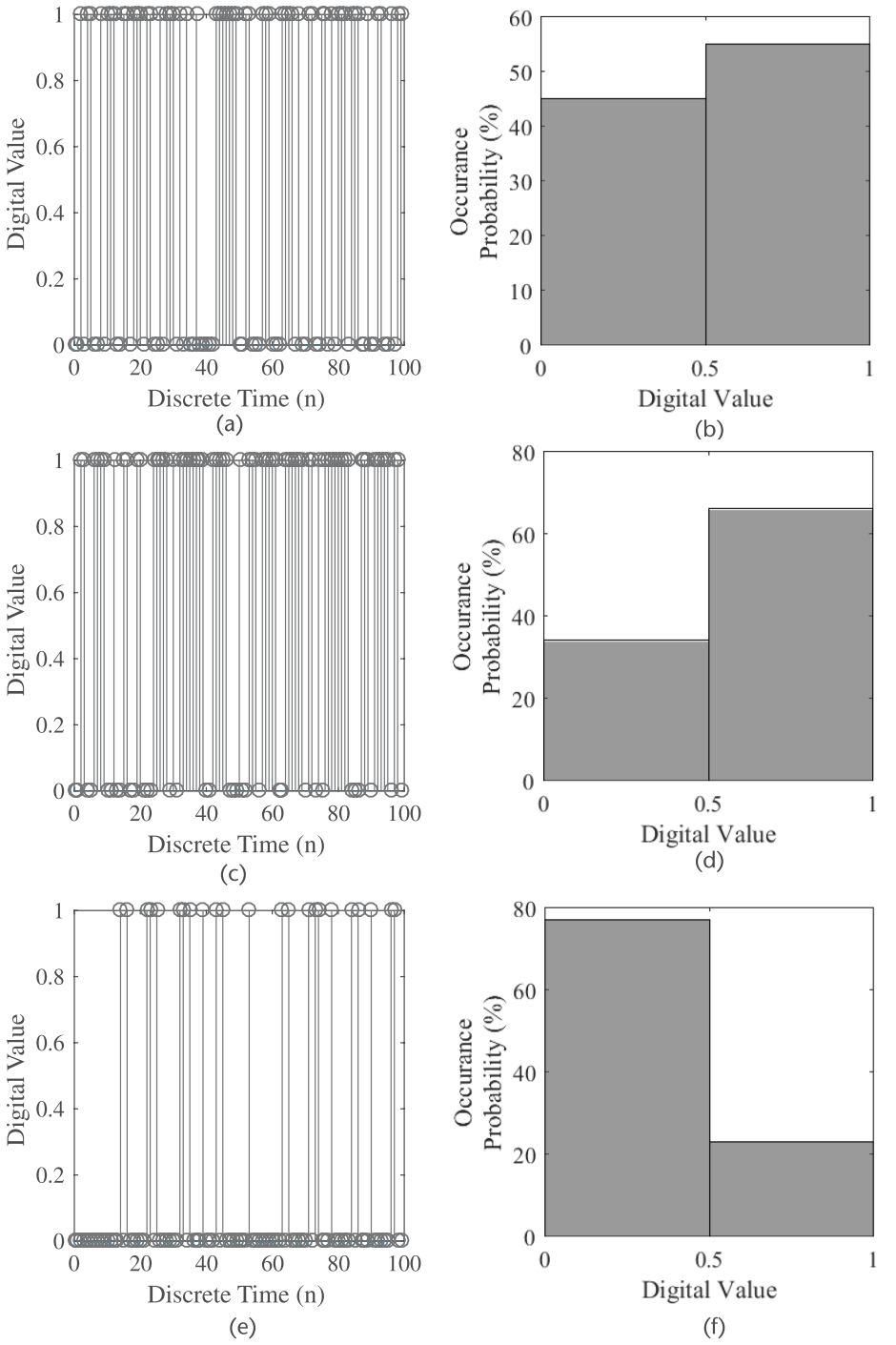

Figure 3.1 Stem plots and histograms of three binary transmissions that each possess different probabilities of ones and zeros being produced. (a) Binary signal (50% ones, 50% zeros), (b) histogram (50% ones, 50% zeros), (c) binary signal (60% ones, 40% zeros), (d) histogram (60% ones, 40% zeros), (e) binary signal (20% ones, 80% zeros), and (f) histogram (20% ones, 80% zeros).

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Suppose we have a random variable X and a real-valued function $g(x)$ that maps it to a new random variable Z; that is, $Z = g(X)$. Since X is actually the mapping of points from a sample space Ω to a collection of real numbers, we are thus performing a mapping of a mapping when solving for the random variable Z; that is, $Z(\omega) = g(X[\omega])$. Consequently, in order to compute the expectation of the random variable Z, namely $E[Z]$, we can employ the following expression:

$$E[Z] = E[g(X)] = \sum_i g(x_i)p_X(x_i), \tag{3.8}$$

which is referred to the expectation of a function of a random variable. One of the most important aspects of (3.8) is that the expectation of Z can be solved using the PMF for X rather than having to determine the PMF of Z. Table 3.2 presents several useful properties associated with the expectation operator.

Table 3.2 Several Useful Properties of Expectation

Name	*Definition*
Linearity	If a and b are deterministic constants and X and Y are random variables, then $E[aX+bY] = E[aX]+E[bY] = aE[X]+bE[Y]$.
Moment	The nth moment ($n \geq 1$) of a real-valued random variable X is defined as the expected value of X raised to the nth power; that is, $\text{Moment}_n(X) = E[X^n]$.
Mean	The mean is the first moment of X; that is, $\text{Moment}_1(X) = E[X] = \mu$.
Variance	The second moment of X with its mean subtracted is its variance; that is, $\text{Moment}_2(X-\mu) = E[(X-\mu)^2] = \text{var}(X) = \sigma^2$.
Central moment	The generalization of the variance is the nth order central moment of X; that is, $E[(X-\mu)^n]$. Note that the *skewness* and *kurtosis* of X are the third-order and fourth-order central moments of X, respectively.
Correlation	The correlation between two random variables X and Y is defined to be equal to $E[XY]$.
Covariance	The covariance between X and Y is defined as $\text{cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$.
Correlation coefficient	The correlation coefficient of two random variables X and Y is given as $\rho_{XY} = E\left[\left(\frac{X-\mu_X}{\sigma_X}\right)\left(\frac{Y-\mu_Y}{\sigma_Y}\right)\right]$.
Markov inequality	If X is a nonnegative random variable, then for any $a > 0$ we have $P(X \geq a) \leq \frac{E[X]}{a}$.
Chebyshev inequality	For any random variable Y and any $a > 0$, we have $P(\|Y\| \geq a) \leq \frac{E[Y^2]}{a^2}$.
Cauchy-Schwarz inequality	The Cauchy-Schwarz inequality states that $\|E[XY]\| \leq \sqrt{E[X^2]E[Y^2]}$, which becomes an equality if and only if X and Y are linearly related.
Independence	The random variables are independent if and only if $E[h(X)k(Y)] = E[h(X)]E[k(Y)]$ for all functions $h(x)$ and $k(y)$.

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3.2 Binary Communication Channels and Conditional Probability

In a digital communication system, we can view the operations of the receiver as attempting to guess what was sent from a particular transmitter when guessing the value of a transmission, whose values can be modeled as a random variable X, and once it has been observed at the receiver, whose values can be modeled by a random variable Y. Consequently, we need some sort of decision rule to figure out which value of X was originally transmitted. This is illustrated in Figure 3.2, where X can output values of either 0 or 1 while Y is observed at the receiver. Notice how the observation Y can be accidentally interpreted as a value other than the one transmitted; that is, $X \neq Y$.

In order to characterize and understand this random transmission environment, we can use the definition for the conditional probability, which is mathematically expressed as

$$\begin{aligned} P(X \in B|Y \in C) &= P(\{X \in B\}|\{Y \in C\}) \\ &= \frac{P(\{X \in B\} \cap \{Y \in C\})}{P(\{Y \in C\})} = \frac{P(X \in B, Y \in C)}{P(Y \in C)}. \end{aligned} \tag{3.9}$$

Additionally, the conditional PMF can also be used to mathematically describe this random phenomenon:

$$\begin{aligned} p_{X|Y}(x_i|y_j) &= P(X = x_i|Y = y_j) = \frac{P(X = x_i, Y = y_j)}{P(Y = y_j)} = \frac{p_{XY}(x_i, y_j)}{p_Y(y_j)} \\ p_{Y|X}(y_j|x_i) &= P(Y = y_j|X = x_i) = \frac{P(X = x_i, Y = y_j)}{P(X = x_i)} = \frac{p_{XY}(x_i, y_j)}{p_X(x_i)} \end{aligned} \tag{3.10}$$

which can be subsequently expressed as

$$p_{XY}(x_i, y_j) = p_{X|Y}(x_i|y_j)p_Y(y_j) = p_{Y|X}(y_j|x_i)p_X(x_i). \tag{3.11}$$

One useful mathematical tool that can be employed to help characterize phenomena described by conditional probabilities and/or conditional PMFs is the law of total probability. For example, suppose that Y is an arbitrary random variable, and we take $A = \{Y \in C\}$, where $C \subset \mathbb{R}$. Consequently, we can define the

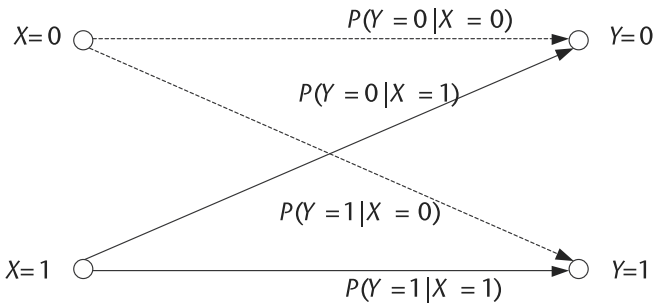

Figure 3.2 An example of a binary channel where transmission values generated by the random variable X are being observed at the receiver as the random variable Y.

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Derive the Markov inequality, $P(X \geq a) \leq \frac{E[X^r]}{a^r}$, using the definition for the expectation.

law of total probability as

$$P(Y \in C) = \sum_i P(Y \in C|X = x_i)P(X = x_i). \tag{3.12}$$

Similarly, if Y is a discrete random variable, taking on distinct values y_i and setting $C = \{y_i\}$, then this yields the following law of total probability:

$$\begin{aligned} P(Y = y_j) &= \sum_i P(Y = y_j|X = x_i)P(X = x_i) \\ &= \sum_i p_{Y|X}(y_j)p_X(x_i). \end{aligned} \tag{3.13}$$

Derive the resulting expressions for the law of total probability for expectation and the substitution law.

Returning to our binary communication channel model described by Figure 3.2, suppose we would like to decide on what values were transmitted based on the observation of the intercepted received signal. In this case, we would like to employ the *maximum a posteriori* (MAP) decision rule, where given an observed $Y = j$, the MAP rule states that we should decide on $X = 1$ if

$$P(X = 1|Y = j) \geq P(X = 0|Y = j), \tag{3.14}$$

and to decide on $X = 0$ otherwise. In other words, the MAP rule decides $X = 1$ if the posterior probability of $X = 1$ given the observation $Y = j$ is greater than the posterior probability of $X = 0$ given the observation $Y = j$. Furthermore, we can observe that

$$P(X = i|Y = j) = \frac{P(X = i, Y = j)}{P(Y = j)} = \frac{P(Y = j|X = i)P(X = i)}{P(Y = j)}, \tag{3.15}$$

which we can then use to rewrite (3.14) as

$$\begin{aligned} \frac{P(Y = j|X = 1)P(X = 1)}{P(Y = j)} &\geq \frac{P(Y = j|X = 0)P(X = 0)}{P(Y = j)} \\ P(Y = j|X = 1)P(X = 1) &\geq P(Y = j|X = 0)P(X = 0). \end{aligned} \tag{3.16}$$

If $X = 0$ and $X = 1$ are equally likely to occur, we can simplify this expression such that it yields the maximum likelihood (ML) rule

$$P(Y = j|X = 1) \geq P(Y = j|X = 0). \tag{3.17}$$

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Code 3.2 Simulate: **random_example.m**

```
% Define simulation parameters
L = 100000; % Transmission length
prob00 = 0.95; % Probability zero received given zero transmitted
prob11 = 0.99; % Probability one received given one transmitted
prob4 = 0.7; % Have 40% 1 values and 60% 0 values

% Create transmitted binary data stream
b4 = round(0.5*rand(1,L)+0.5*prob4);
                              % Have 40% 1 values and 60% 0 values
b4hat = b4; % Initialize receive binary data stream

% Randomly select 1 and 0 values for flipping
ind_zero = find(b4 == 0); % Find those time instances with zero values
ind_one = find(b4 == 1); % Find those time instances with one values
ind_flip_zero = find(round(0.5*rand(1,length(ind_zero))
                  +0.5*(1-prob00)) == 1); % Flip zero bits to one bits
ind_flip_one = find(round(0.5*rand(1,length(ind_one))
                  +0.5*(1-prob11)) == 1); % Flip one bits to zero bits

% Corrupt received binary data stream
b4hat(ind_zero(ind_flip_zero)) = 1; % Flip 0 to 1
b4hat(ind_one(ind_flip_one)) = 0; % Flip 1 to 0

% Calculate bit error statistics
b4error_total = sum(abs(b4-b4hat))/L;
b4error_1 = sum(abs(b4(ind_one) - b4hat(ind_one)))/length(ind_one);
b4error_0 = sum(abs(b4(ind_zero) - b4hat(ind_zero)))/length(ind_zero);
```

Furthermore, in the general case, we can rearrange (3.16) such that it yields the likelihood ratio; namely:

$$\frac{P(Y=j|X=1)}{P(Y=j|X=0)} \geq \frac{P(X=0)}{P(X=1)} \tag{3.18}$$

where the right-handed side is referred to as the threshold since it does not depend on j.

Given this mathematical formulation, let us now work with this same example via computer simulation. Using the following MATLAB script, we can model a binary channel where we produce `L` binary values with a probability of `prob4` being one values and the rest being zero values. Furthermore, we assume that the binary channel being used is not symmetric, meaning that the probability of one values being flipped into zero values is different than the probability of zero values being flipped into one values. Note that the flipping of bit values is considered to be an error produced by the channel. Consequently, we define the probability of a transmitted one value being received as a one value to be equal to `prob11` while the probability of a transmitted zero value being received as a zero value is equal to `prob00`.

One of fundamental metrics for assessing the performance of any digital communication system is the *probability of bit error*, or the bit error rate (BER). The BER characterizes the amount of bit errors received relative to the total bits transmitted. For various applications, different BER values are considered

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acceptable while others are considered intolerable. For instance, for typical wireless data transmission applications, a BER of 10^{-5} is considered an acceptable amount of error introduced into the transmission.

In the MATLAB example above involving the binary channel, we would like to characterize the BER values to see if they conform to the parameters we defined. In Figure 3.3, we have the BER for the overall transmission, as well as for only the the transmissions of one values and of zero values. Notice how the BER for the one values only corresponds to the complement of the probability `prob11`, while the same can be observed for the BER of the zero values only and it being the complement of the probability `prob00`. Remember that in order to obtain an accurate statistical assessment of the BER, a very large number of binary values need to be generated.

3.3 Modeling Continuous Random Events in Communication Systems

As we have seen earlier in this chapter, it is possible to mathematically compute the probability of a random event occurring within a communication system described by a discrete random variable; namely,

$$P(a \leq X < b) = \sum_{i=a}^{b-1} p_X(i), \tag{3.19}$$

where X is the discrete random variable, and both a and b are boundaries of a subset belonging to the sample space Ω. However, suppose now that X is a *continuous random variable*, which can take on the entire continuum of values within the interval (a, b). In order to compute the same probability in this case, we can start by realizing this scenario as the summation of an infinite number of points in (a, b) with the space between samples equal to Δx.

There are numerous random elements contained within a communication system where each can produce a continuum of possible output values. As we will discuss later in this chapter, one of these elements represented by a continuous random variable is the noise introduced in a transmission channel. Suppose this noise can be represented by an infinite number of samples such that our Δx becomes

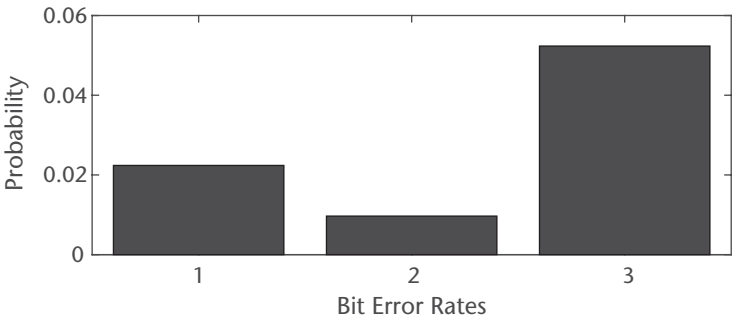

Figure 3.3 Binary channel error probabilities when the probability for a zero being received uncorrupted is 0.95 and the probability for a one being received uncorrupted is 0.99. Note that the transmission consists of 40% ones and 60% zeros. 1 = Total, 2 = one transmitted, 3 = zero transmitted.

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so tiny the Δx ultimately converges to dx and the summation in (3.19) becomes an integral expression. Therefore, we can express the probability of a continuous random variable modeling the noise producing output values ranging between a and b using the expression

$$P(a \leq X < b) = \int_{a}^{b} f(t)dt, \tag{3.20}$$

where $f(t)$ is called the *probability density function* (PDF). Note that the PDF is the continuous version of the PMF that we discussed previously in this chapter. Moreover, generally we can express the probability of a continuous random variable using the PDF by the expression

$$P(X \in B) = \int_{B} f(t)dt = \int_{-\infty}^{+\infty} I_B(t)f(t)dt, \tag{3.21}$$

where $I_B(t)$ is an indicator function that produces an output of unity whenever a value of t belongs to the set B and produces an output of zero otherwise. Note that $\int_{-\infty}^{+\infty} f(t)dt = 1$, $f(t)$ is nonnegative, $f(t)$ approximately provides the probability at a point t, and

$$P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < X < b), \tag{3.22}$$

where the end points do not affect the resulting probability measure. Finally, a summary of several commonly used PDFs are presented in Table 3.3, including uniform, Gaussian, and exponential random variables. Note that Gaussian random variables are often used to model the randomness of the noise introduced in a communication channel, while the exponential random variable is used in medium access protocols to help provide random back-off times in the event that two or more wireless systems are attempting to communicate over the same wireless channel via a contention-based framework.

Similar to the expectation of a single discrete random variable, the expectation for a continuous random variable X with PDF $f(x)$ can be computed using the following expression:

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x)f(x)dx, \tag{3.23}$$

where $g(.)$ is some real function that is applied to the random variable X.

Many random variables of practical interest are not independent, where it is often the case that the outcome of one event may depend on or be influenced by the result of another event. Consequently, it is sometimes necessary to compute the *conditional probability* and *conditional expectation*, especially in circumstances where we have to deal with problems involving more than one random variable.

Unlike the conditional probability for a discrete random variable, the conditional probability for a continuous random variable needs to defined in an alternative manner since the probability of a single exact point occurring is zero;

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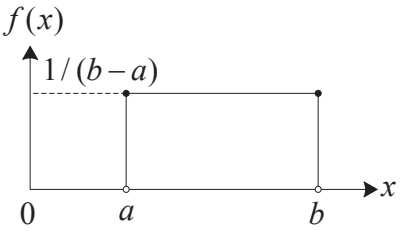
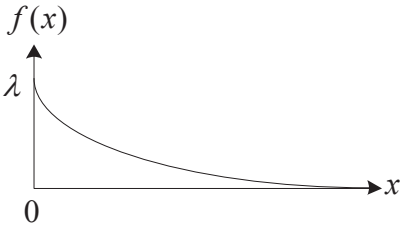
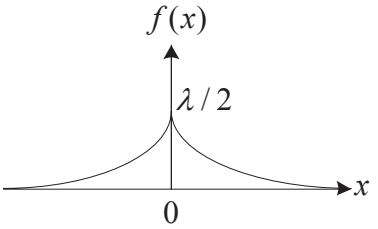
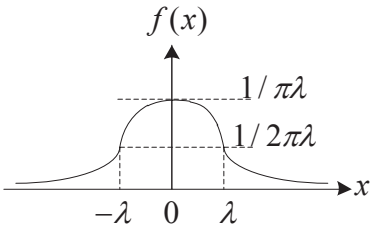
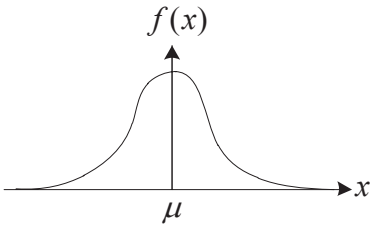
Table 3.3 Several Frequently Used Probability Density Functions

Random Variable	*PDF Definition*	*Graphical Representation*
Uniform	$f(x) = \begin{cases} \frac{1}{(b-a)}, & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$	
Exponential	$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \ge 0 \\ 0, & x < 0 \end{cases}$	
Laplace	$f(x) = \frac{\lambda}{2} e^{-\lambda \lvert x \rvert}$	
Cauchy	$f(x) = \frac{\lambda/\pi}{\lambda^2 + x^2}$, $\lambda > 0$	
Gaussian	$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-0.5((x-\mu)/\sigma)^2}$	

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that is, $P(X = x) = 0$. As a result, if we employ the definition for the conditional probability for a discrete random variable from (3.9); namely,

$$P(Y \in C|X = x) = \frac{P(Y \in C, X = x)}{P(X = x)}, \tag{3.24}$$

we observe that the occurrence of $P(X = x) = 0$ would yield a divide-by-zero scenario. Consequently, we need to determine another definition for the conditional probability (and conditional expectation) that would work within the continuous random variable framework.

It can be shown that in order to calculate the conditional probability, one must employ a *conditional density* [1], which can be defined as

$$f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)}, \tag{3.25}$$

where $f_X(x) > 0$. Thus, leveraging the conditional density, we can now compute the conditional probability without concern of a divide-by-zero scenario by solving for the following expression:

$$P(Y \in C|X = x) = \int_C f_{Y|X}(y|x)dy. \tag{3.26}$$

Furthermore, we can define the law of total probability as the following:

$$P(Y \in C) = \int_{-\infty}^{+\infty} P(Y \in C|X = x)f_X(x)dx, \tag{3.27}$$

where we weigh all the conditional probabilities by the PDF of X before integrating them together to form the overall probability of the event $Y \in C$. Finally, just as in the discrete random variable case, we can employ a form of substitution law for continuous random variables when dealing with conditional probability, which is defined by

$$P((X, Y) \in A|X = x) = P((x, Y) \in A|X = x). \tag{3.28}$$

Note that if X and Y are independent, then the joint density factors, yielding the following expression for the conditional density:

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{f_{XY}(x, y)}{f_X(x)} \\ &= \frac{f_X(x)f_Y(y)}{f_X(x)} \\ &= f_Y(y), \end{aligned} \tag{3.29}$$

which implies that when the two random variables are independent, we do not need to condition one event on the other.

Similarly, the conditional expectation when dealing with continuous random variables is defined as the following expression employing the conditional density;

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namely,

$$E[g(Y)|X = x] = \int_{-\infty}^{+\infty} g(y) f_{Y|X}(y|x) dy. \tag{3.30}$$

Furthermore, the law of total probability for a conditional expectation is given as

$$E[g(X, Y)] = \int_{-\infty}^{+\infty} E[g(X, Y)|X = x] f_X(x) dx, \tag{3.31}$$

and the substitution law for a conditional expectation is defined as

$$E[g(X, Y)|X = x] = E[g(x, Y)|X = x]. \tag{3.32}$$

3.3.1 Cumulative Distribution Functions

For both PDFs and PMFs of random variables modeling random elements within a communication system, it is sometimes important to visualize the *cumulative distribution function* or CDF, especially since it provides another perspective on how the random variable behaves probabilistically. Furthermore, the CDF can sometimes be use to solve problems that would otherwise be difficult to access via some other definition.

Mathematically speaking, we can define the CDF by the following expression:

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f(t) dt, \tag{3.33}$$

which describes the probability that the outcome of an experiment described by the random variable X is less than or equal to the dummy variable x.

As an example, suppose that we want to calculate the probability of $P(a \leq X < b)$ using the PDF shown in Figure 3.4(a). One approach for quickly evaluating this probability is to leverage the *tail probabilities* of this distribution; namely, $P(X < a)$ (shown in Figure 3.4[b]) and $P(X < b)$ (shown in Figure 3.4[c]). Notice how the tail probabilities are actually the CDFs of X based on (3.33), where $F_X(a) = P(X < a)$ and $F_X(b) = P(X < b)$. Consequently, given that we are only interested in the region of the PDF where these two tail probabilities do not intersect, we can compute the following probability:

$$P(a \leq X < b) = P(X < b) - P(X < a) = F_X(b) - F_X(a), \tag{3.34}$$

where all we really need are the values for the CDF of X at $x = a$ and $x = b$.

Several fundamental characteristics of the CDF include the fact that $F_X(x)$ is bounded between zero and one, and that $F_X(x)$ is a nondecreasing function; that is, $F_X(x_1) \leq F_X(x_2)$ if $x_1 \leq x_2$. Furthermore, the PDF is the derivative of the CDF in terms of the dummy variable x, which we can define as:

$$f_X(x) = \frac{d}{dx} F_X(x). \tag{3.35}$$

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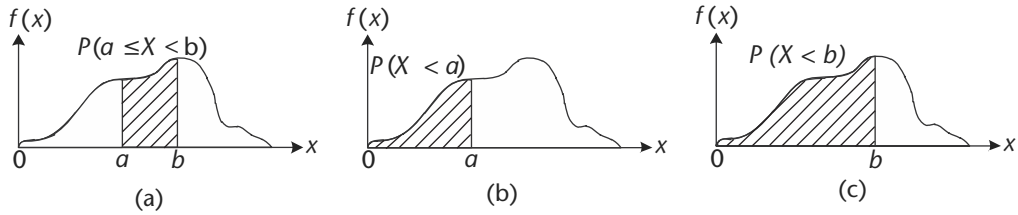

Figure 3.4 An example of how the CDF can be used to obtain the tail probabilities $P(X < a)$ and $P(X < b)$ in order to quickly calculate $P(a \leq X < b)$. (a) The region of the PDF of the random variable X that needs to be integrated in order to yield $P(a \leq X < b)$, (b) the region of the PDF of the random variable X that needs to be integrated in order to yield $P(X < a)$, and (c) the region of the PDF of the random variable X that needs to be integrated in order to yield $P(X < b)$.

The *Q function* is a convenient way to express right-tail probabilities for Gaussian random variables, $P(X > x)$. Mathematically, this is equivalent to finding the complementary CDF of X; namely [2]:

$$Q(x) = 1 - F_X(x) = 1 - P(X \leq x)$$

$$= P(X > x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} dt,$$

where $F_X(x)$ is the CDF of X.

One important use for CDFs is having them define the exact probabilistic nature of a random element within a communication system. Noise generation, binary outputs of an information source, and random access of a wireless channel by multiple users can all be characterized exactly using CDFs. Consequently, when modeling these phenomena in a computer simulation, we use a RNG that is defined by one of these CDFs. In the MATLAB computer simulation environment, there exists a variety of RNGs that can be used in communication systems experiments, including those based on uniform, Gaussian (normal), and Rayleigh random variables. These random variables can be generated in MATLAB via the `rand` and `randn` functions, as well as their combination to create other random variables. For example, the MATLAB code in Code 3.3 produces three vectors of random values generated in such a way that they possess statistical characteristics equaivalent to the uniform, Gaussian, and Rayleigh random variables. Furthermore, using the randomly generated values using these RNGs, it is possible for us to determine the probability densities such that we can then generate their cummulative distribution functions as well as calculate the probability that these random variables produce a value between 0.7 and 1.0.

To obtain values that possess uniform and Gaussian distributions in MATLAB, one can simply use the `rand` and `randn` functions. If a very large number of these random values are generated, it is possible to observe the uniform and Gaussian PDFs, as shown in Figures 3.5(a) and 3.5(c). Since the cumulative distribution function (CDF) is the progressive accumulation of the PDFs from negative infinity to positive infinity, those can also be readily generated from the PDF data, as shown

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Code 3.3 Information Source: **random_example.m**

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Random Variable PDFs and CDFs

% Define simulation parameters
L = 1000000; % Length of random samples
mean_normal = 1; stddev_normal = 2;
                    % Mean and standard deviation of normal RV values
res_hist = 100; % Histogram resolution

% Generate random samples for different distributions
b_unif = rand(1,L); % Uniform random values
b_normal = (stddev_normal*randn(1,L) + mean_normal);
                    % Normal random values
b_rayleigh = (sqrt(randn(1,L).^2 + randn(1,L).^2));
                    % Rayleigh random values

% Obtain cumulative distribution functions
[N_unif, edges_unif] = histcounts(b_unif,res_hist);
N_unif_cum = cumsum(N_unif)./L;
[N_normal, edges_normal] = histcounts(b_normal,res_hist);
N_normal_cum = cumsum(N_normal)./L;
[N_rayl, edges_rayl] = histcounts(b_rayleigh,res_hist);
N_rayl_cum = cumsum(N_rayl)./L;

% Calculate probability of values between 0.7 and 1
x_lower = 0.7;
x_upper = 1.0;
unif_ind_range = find((x_lower <= edges_unif)
                            & (edges_unif < x_upper));
normal_ind_range = find((x_lower <= edges_normal)
                            & (edges_normal < x_upper));
rayl_ind_range = find((x_lower <= edges_rayl)
                            & (edges_rayl < x_upper));
prob_unif = sum(N_unif(unif_ind_range))./L;
prob_normal = sum(N_normal(normal_ind_range))./L;
prob_rayl = sum(N_rayl(rayl_ind_range))./L;
```

for the uniform and Gaussian random values in Figures 3.5(b) and 3.5(d). As for the values produced by a Rayleigh random variable, a quick way of producing these values is to take two *independently and identically distributed* (i.i.d.) Gaussian random variables, take the square of both values, sum them together, and then take their square root. As a result of this operation, and given a very large number of values generated, it is possible to create a PDF and a CDF of a Rayleigh random variable as shown in Figures 3.5(e) and 3.5(f). Note that if one wants to find the probability of a randomly generated value produced by these functions between 0.7 and 1.0, simply either sum up the density values between this range or take the CDF values at these end points and subtract them from each other.

3.4 Time-Varying Randomness in Communication Systems

Until now we have been exploring how to model random phenomena where these probabilistic characteristics remain the same throughout time. Although this

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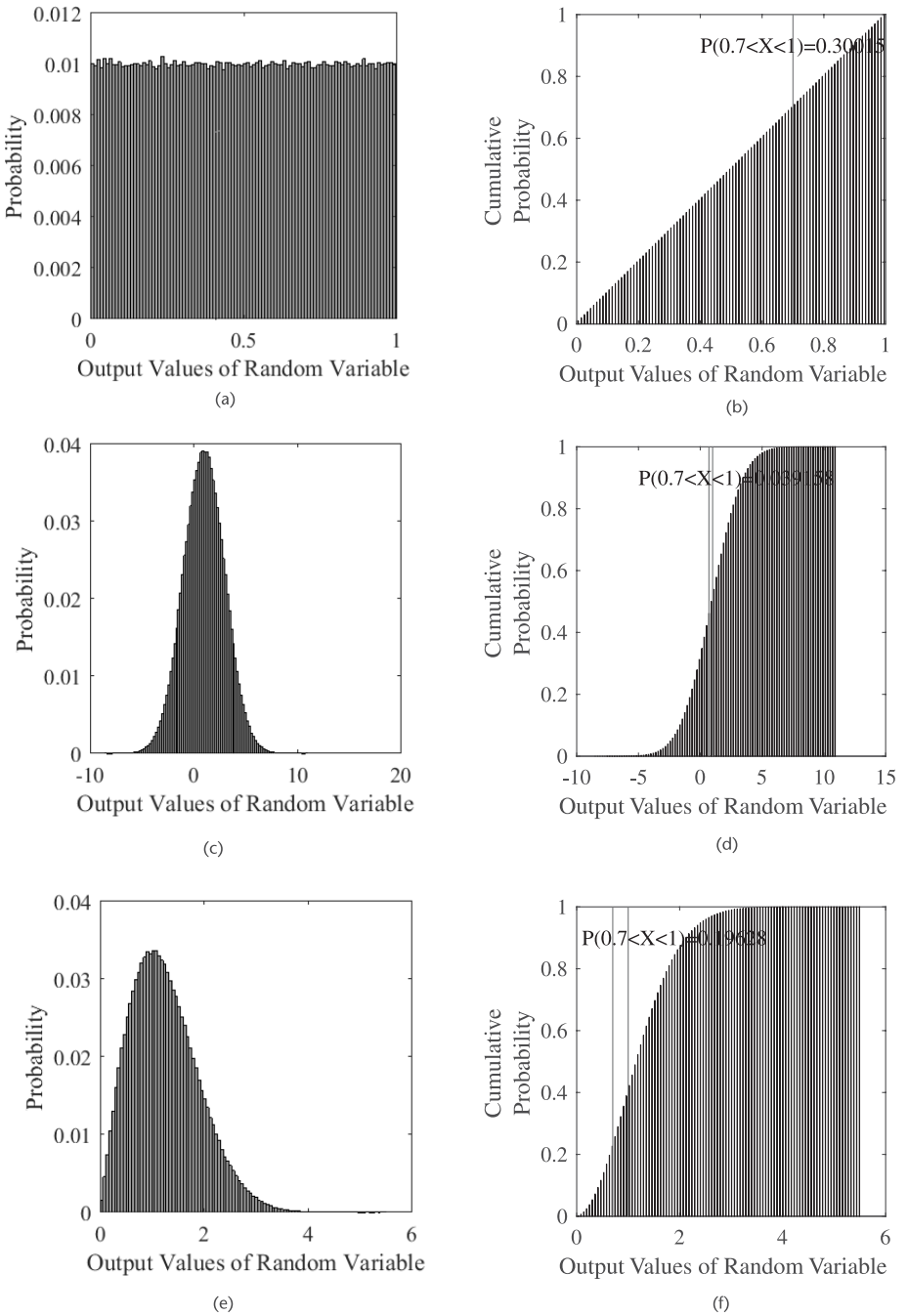

Figure 3.5 Various cumulative distribution functions and associated probability density functions. (a) Uniform PDF, (b) uniform CDF, (c) Gaussian PDF, (d) Gaussian CDF, (e) Rayleigh PDF, and (f) Rayleigh CDF.

simplifies the mathematical derivation of these models, this may not accurately describe the random phenomena. For example, the binary output values from an information source might change over time depending on the real-world data being encoded, such as security camera footage of a dynamic urban environment

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or the internet traffic of a human user on a computer. Consequently, it is necessary to develop a more comprehensive mathematical representation of these random phenomena that are also functions of time. We refer to these representations as *random processes* or *stochastic processes*. A random process is a family of time domain functions depending on the parameters t and ω; that is, we can define a random process by the function:

$$X(t) = X(t, \omega), \tag{3.36}$$

where the left-handed side is the shortened representation of a random process that implicitly assumes the existence of a mapping of an outcome ω from the sample space Ω to a real-valued number. Note that the domain of ω is Ω while the domain of t is either $\mathbb{R}$ for continuous-time random processes or $\mathbb{Z}$ for discrete-time random processes. An illustration depicting how a random process consists of a family of time domain functions is shown in Figure 3.6.

Suppose we have a random process that is noncountable infinite for each time instant t. Given this situation, we can define its first-order distribution $F(x, t)$ and first-order density $f(x, t)$ as

$$F(x, t) = P(X(t) \leq x) \tag{3.37}$$

and

$$f(x, t) = \frac{\partial F(x, t)}{\partial x}. \tag{3.38}$$

For determining the statistical properties of a random process, knowledge from the function $F(x_1, \ldots, x_n; t_1, \ldots, t_n)$ is required. However, in most communication system applications only certain averages are actually needed. For instance, one of the mostly commonly used statistical characterizations for a random process is the *mean function*, where the mean function $\mu_X(t)$ of a random process $X(t, \omega)$ is the expectation of the random process at a specific time instant t. This can be mathematically represented by

$$\mu_X(t) = E[X(t, \omega)]. \tag{3.39}$$

Another useful statistical characterization tool for a random process $X(t, \omega)$ is the *autocorrelation function* $R_{XX}(t_1, t_2)$, where we evaluate the amount of

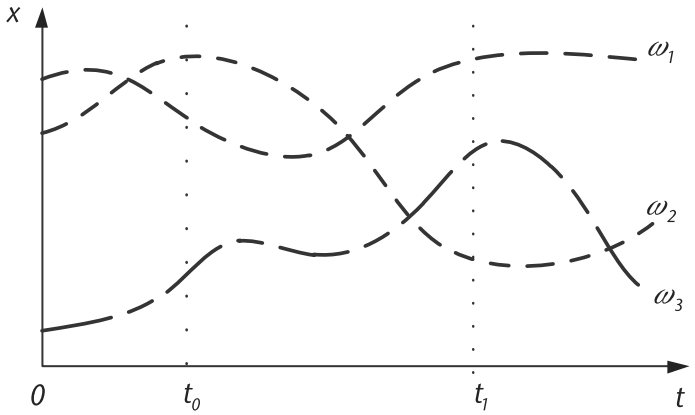

Figure 3.6 Illustration of a random process $X(t, \omega)$.

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correlation that the random process $X(t,\omega)$ possesses at two different time instants t_1 and t_2. We can define this mathematically by the expression

$$\begin{aligned} R_{XX}(t_1,t_2) &= E[X(t_1,\omega)X^*(t_2,\omega)] \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2^* f(x_1,x_2;t_1,t_2)dx_1dx_2. \end{aligned} \tag{3.40}$$

Note that the value of the diagonal for $R_{XX}(t_1,t_2)$ is the average power of $X(t,\omega)$; namely,

$$E[|X(t,\omega)|^2] = R_{XX}(t,t). \tag{3.41}$$

Several other useful properties and observations about the autocorrelation function include the following:

1. Since $R_{XX}(t_1,t_2) = E[X(t_1,\omega)X^*(t_2,\omega)]$, then $R_{XX}(t_2,t_1) = E[X(t_2,\omega)X^*(t_1,\omega)] = R_{XX}^*(t_1,t_2)$.
2. We have $R_{XX}(t,t) = E[|X(t,\omega)|^2] \geq 0$.
3. A random process for which $E[|X(t,\omega)|^2] < \infty$ for all t is called a *second-order process*.
4. For $R_{XX}(t,t) = E[|X(t,\omega)|^2] \geq 0$ and given time instants t_1 and t_2, we have the following inequality:

$$|R_{XX}(t_1,t_2)| \leq \sqrt{E[|X(t_1,\omega)|^2]E[|X(t_2,\omega)|^2]}.$$

5. A normalized process occurs when $X(t,\omega)/\sqrt{C_{XX}(t,t)}$.

An extension of the definition for the autocorrelation function is the *autocovariance function* $C_{XX}(t_1,t_2)$ of $X(t,\omega)$, which is the covariance of the random process $X(t,\omega)$ at time instants t_1 and t_2. Mathematically, we can represent the autocovariance function by the expression.

$$C_{XX}(t_1,t_2) = R_{XX}(t_1,t_2) - \mu_X(t_1)\mu_X^*(t_2). \tag{3.42}$$

Note that for $t_1 = t_2$, the autocovariance function produces the variance of $X(t,\omega)$. Furthermore, we can sometimes represent the autocovariance function of a random process $X(t,\omega)$ using a normalized metric called the correlation coefficient, which we define as

$$\rho_{XX}(t_1,t_2) = \frac{C_{XX}(t_1,t_2)}{\sqrt{C_{XX}(t_1,t_1)C_{XX}(t_2,t_2)}}. \tag{3.43}$$

3.4.1 Stationarity

Although random processes may possess a significant amount variability across time, there does exist a subset of random processes that exhibit the same behavior at any two time instants; that is, the random process is time-invariant. We refer to these types of random processes as *stationary processes*. Two common forms of stationary processes are *strict-sense stationary* (SSS) random processes and *wide-sense stationary* (WSS) random processes. A random process is SSS whenever its statistical properties are invariant to a shift of the origin; that is, the random process

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$X(t, \omega)$ and $X(t+c, \omega)$ both possess the same statistics for any time shift c. Therefore, the nth-order density of an SSS random process would be equal to, by definition, the following expression:

$$f(x_1, \ldots, x_n; t_1, \ldots, t_n) = f(x_1, \ldots, x_n; t_1 + c, \ldots, t_n + c) \tag{3.44}$$

for any time shift c.

It follows that $f(x; t) = f(x; t + c)$ for any time shift c, which means that the first-order density of $X(t, \omega)$ is independent of the time t; namely,

$$f(x; t) = f(x). \tag{3.45}$$

Furthermore, $f(x_1, x_2; t_1, t_2) = f(x_1, x_2; t_1 + c, t_2 + c)$ is independent of c for any value of c. Consequently, this means that the density becomes

$$f(x_1, x_2; t_1, t_2) = f(x_1, x_2; \tau), \text{ where } \tau = t_1 - t_2. \tag{3.46}$$

Thus, the joint density of the random process at time instants t and $t + \tau$ is independent of t and is equal to $f(x_1, x_2; \tau)$.

Although SSS random processes can yield mathematically tractable solutions based on their useful time-invariant property, the occurrence of SSS random processes in actual communication systems is not very frequent. On the other hand, the WSS random processes occur more frequently in the analyses of communication systems. A random process $X(t, \omega)$ is considered to be WSS whenever both of the following properties are true:

- The mean function $\mu_X(t)$ does not depend on time t; that is, $\mu_X(t) = E[X(t, \omega)] = \mu_X$.
- The autocorrelation function $R_{XX}(t + \tau, t)$ only depends on the relative difference between t and $t + \tau$; that is, $R_{XX}(t + \tau, t) = R_{XX}(\tau)$.

Several observations about WSS random processes include the following:

- The average power of a WSS random process is independent of time since $E[|X(t, \omega)|^2] = R_{XX}(0)$.
- The autocovariance function of a WSS random process is equal to $C_{XX}(\tau) = R_{XX}(\tau) - |\mu_X|^2$.
- The correlation coefficient of a WSS random process is given by $\rho_{XX}(\tau) = C_{XX}(\tau)/C_{XX}(0)$.
- Two random processes $X(t, \omega)$ and $Y(t, \omega)$ are jointly WSS if each is WSS and their cross-correlation depends on $\tau = t_1 - t_2$.
- If the random process $X(t, \omega)$ is WSS and uncorrelated, then $C_{XX}(\tau) = q\delta(\tau)$, where q is some multiplicative constant.

There exists another form of stationarity characteristic that often occurs in wireless data transmission. A *cyclostationary* random process $Y(t)$ is defined by a mean function $\mu_Y(t)$ that is periodic across time t as well as an autocorrelation function $R_{YY}(\tau + \theta, \theta)$ that is periodic across θ for a fixed value of τ. Consequently, a cyclostationary random process $Y(t)$ with period T_0 can be

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described mathematically by

$$\bar{R}_{YY}(\tau) = \frac{1}{T_0}\int_0^{T_0} R_{XX}(\tau+\theta,\theta)d\theta. \tag{3.47}$$

In the area of communication systems engineering, cyclostationary random processes are often leveraged in the detection of wireless signals in noisy channel environments, where a target wireless signal will produce a unique characteristic function that will enable its identification assuming that multiple signals are present within the same spatiotemporal-frequency region.

3.5 Gaussian Noise Channels

As mentioned previously, Gaussian; that is, normal, random variables have often been used to model the noise introduced within a communication channel. In fact, many analyses of communication systems and their performance are often conducted assuming that the noisy channel possess Gaussian random behavior. Consequently, this makes the Gaussian random variable one of the most frequently used random variable in the study of communication systems.

We define the univariate Gaussian PDF as

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}, \tag{3.48}$$

where μ is the mean of the Gaussian random variable X, and σ^2 is the variance of X. In the case that $\mu = 0$ and $\sigma^2 = 1$, we refer to X as a standard normal random variable.

Although the univariate Gaussian distribution is frequently used in numerous applications, there are several instances where we must employ the *bivariate Gaussian distribution* in order to characterize a specific application involving two Gaussian random variables possessing some degree of correlation between each other; for example, complex baseband transmission channel with inphase and quadrature signal components. An illustration of an example of a bivariate Gaussian distribution is shown in Figure 3.7.

Mathematically speaking, the general definition for a bivariate Gaussian density with parameters μ_X, μ_Y, σ_X^2, σ_Y^2, and correlation coefficient ρ is given by

$$f_{XY}(x,y) = \frac{\exp\left(\frac{-1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right)\right)}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}, \tag{3.49}$$

where the correlation coefficient is defined as

$$\rho = E\left[\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)\right]. \tag{3.50}$$

Suppose we would like to model a complex baseband channel where the inphase and quadrature noise contributions are represented by bivariate Gaussian random variables. In several instances, we would like to have the inphase and quadrature

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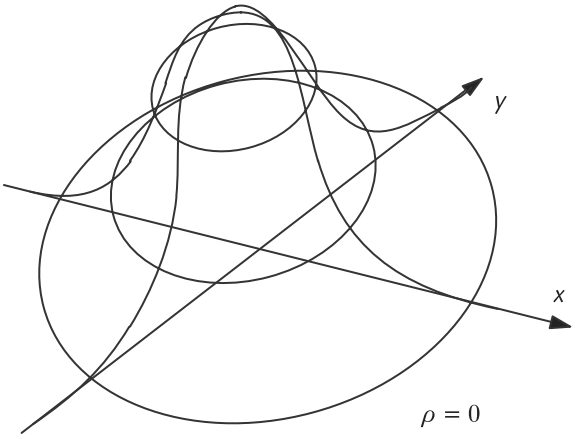

Figure 3.7 Density of a bivariate normal distribution with no correlation; that is, $\rho = 0$.

Code 3.4 Information Source: **`random_example.m`**

```
% Define simulation parameters
L = 1000000; % Length of data streams
res_hist = 100; % Histogram resolution
std_dev = 5; % Standard deviation of input Gaussian variables

% Create uncorrelated 2D Gaussian random variable
x_normal_1 = std_dev.*randn(1,L);
y_normal_1 = std_dev.*randn(1,L);

% Create correlated 2D Gaussian random data stream
x_normal_2 = x_normal_1+0.1*y_normal_1;
y_normal_2 = y_normal_1+0.9*x_normal_1;
```

components of this noise to be uncorrelated. In other situations, we might want them to be very correlated. As a result, we need to make sure we model these bivariate Gaussian random values accurately. The MATLAB script in Code 3.4 models these types of bivariate Gaussian random variables representing channel noise, where we have employed the function `randn` in order to generate two vectors of length `L` that contain values possessing Gaussian characteristics. Since we have generated these vectors separately, by default they are uncorrelated with each other in this situation. Furthermore, we have made both vectors to be zero mean, and they both possess standard deviations of `std_dev`. From these two vectors, we can readily obtain an uncorrelated bivariate Gaussian distributed data as well as a correlated bivariate Gaussian distributed data.

Based on the two-dimensional density functions shown in Figure 3.8, we can readily observe the difference between uncorrelated and correlated bivariate Gaussian random variables. In the uncorrelated case, the bivariate Gaussian random variables appear to be symmetric about the *x*/*y* plane, which can be seen in Figure 3.8(a) (3-dimensional viewpoint) and Figure 3.8(b) (top-down viewpoint). However, once we introduce some correlation between the inphase and quadrature components of the bivariate Gaussian random variables, this begins to warp the shape of the density, compressing it in one direction while expanding it in another.

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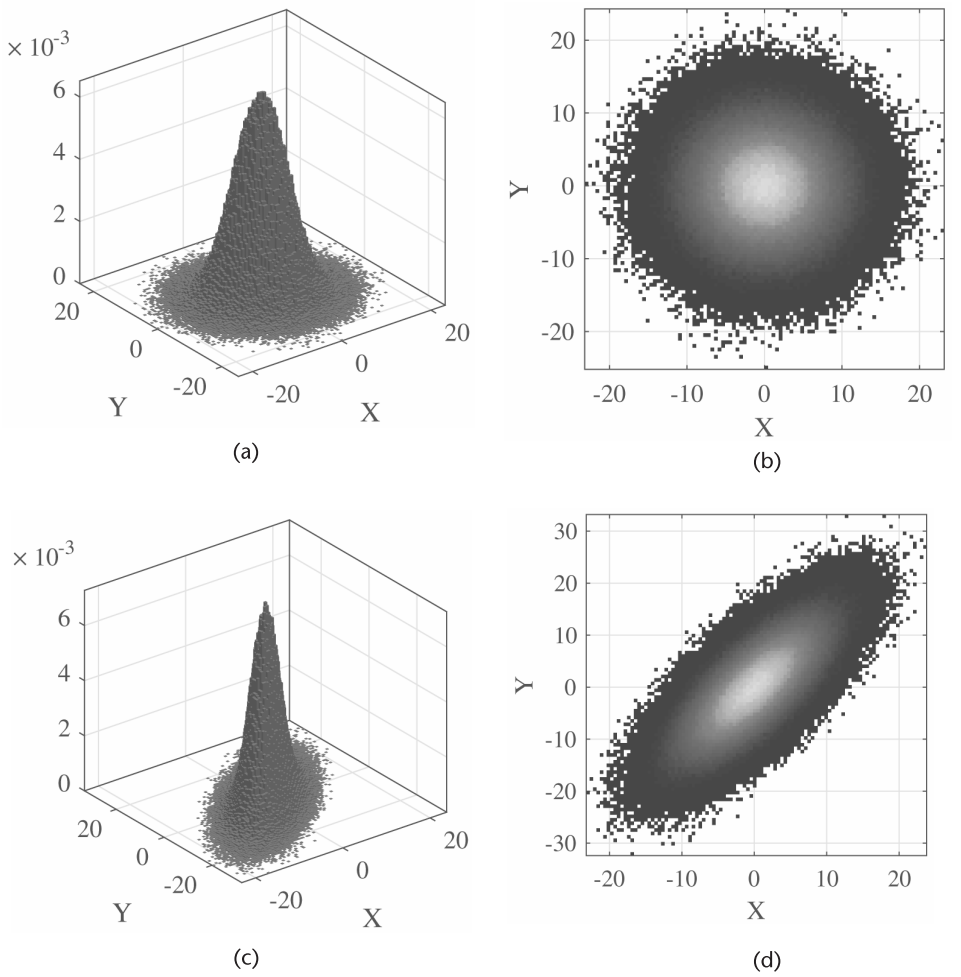

Figure 3.8 Several examples of correlated and uncorrelated bivariate Gaussian densities. (a) Uncorrelated bivariate Gaussian (3-D view), (b) uncorrelated bivariate Gaussian (top-down view), (c) correlated bivariate Gaussian (3-D view), and (d) correlated bivariate Gaussian (top-down view).

This can be observed in Figure 3.8(c) (3-dimensional viewpoint) and Figure 3.8(d) (top-down viewpoint).

3.5.1 Gaussian Processes

As mentioned before, situations exist where the probability characteristics of a random phenomena representing an element of a communication system varies over time (e.g., the properties of a noisy channel represented by a Gaussian random variable). Consequently, we need a mathematical representation that can account for the time-dependent randomness of these phenomena, especially for those events modeled by Gaussian random variables. Refering to Section 3.4, we can model a time-varying Gaussian random variable by a *Gaussian process*, which is a stochastic process whose realizations consist of random values associated with every point in a range of times (or of space) such that each such random variable has a normal distribution. Moreover, every finite collection of those random variables has a multivariate normal distribution.

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Gaussian processes are important in statistical modeling because of properties inherited from the normal distribution. For example, if a random process is modeled as a Gaussian process, the distributions of various derived quantities can be obtained explicitly. Such quantities include the average value of the process over a range of times and the error in estimating the average using sample values at a small set of times.

Given the following expression:

$$y = \int_0^T g(t)X(t)\,dt \tag{3.51}$$

we can say that $X(t)$ is a Gaussian process if

- $E(y^2)$ is finite (i.e., does not blow up).
- Y is Gaussian-distributed for every $g(t)$.

Note that the random variable Y has a Gaussian distribution, where its PDF is defined as

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} e^{\frac{-(y-\mu_Y)^2}{2\sigma_Y{}^2}}, \tag{3.52}$$

where μ_Y is the mean and $\sigma_Y{}^2$ is the variance. Such processes are important because they closely match the behavior of numerous physical phenomena, such as *additive white Gaussian noise* (AWGN).

Why is an uncorrelated random process referred to as white, such as in the case of additive white Gaussian noise?

3.6 Power Spectral Densities and LTI Systems

To analyze a communication system in the frequency domain, the power spectral density (PSD), $S_{XX}(f)$, is often used to characterize the signal, which is obtained by taking the Fourier transform of the autocorrelation $R_{XX}(\tau)$ of the WSS random process $X(t)$. The PSD and the autocorrelation of a function, $R_{XX}(\tau)$, are mathematically related by the *Einstein-Wiener-Khinchin* (EWK) relations; namely,

$$S_{XX}(f) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j2\pi f\tau}\,d\tau \tag{3.53}$$

$$R_{XX}(f) = \int_{-\infty}^{\infty} S_{XX}(\tau)e^{+j2\pi f\tau}\,df \tag{3.54}$$

A very powerful consequence of the EWK relations is its usefulness when attempting to determine the autocorrelation function or PSD of a WSS random process that is the output of an LTI system whose input is also a WSS random process. Specifically, suppose we denote $H(f)$ as the frequency response of an LTI system $h(t)$. We can then relate the power spectral density of input and output

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random processes by the following equation:

$$S_{YY}(f) = |H(f)|^2 S_{XX}(f), \tag{3.55}$$

where $S_{XX}(f)$ is the PSD of input random process and $S_{YY}(f)$ is the PSD of output random process. This very useful relationship is illustrated in Figure 3.9.

To understand the impact of an LTI system on the PSD of a random process, we can use the MATLAB script in Code 3.5, where we have a bivariate uniform random number generator producing inphase and quadrature values. To highlight the impact of LTI systems on PSDs, we have designed two filters using the `firls` function. One of the filters has a relatively large passband while the other filter has a relatively small passband. The bivariate uniform random values are then filtered by both systems and we can observe the resulting PSDs in Figure 3.10.

The three dimensional and top-down perspectives of the original bivariate uniform random values are shown in Figures 3.10(a) and 3.10(b). We observe that the density almost appears to be a rectangular block, which is what we are expecting from a bivariate uniform. Once we filter this data using the filter with the narrow passband region and observe the resulting PSD, we can readily notice the effects of the filtering operation, with most of the PSD being filtered away at the peripherals. This is evident in the three-dimensional and top-down perspectives of these filtered bivariate uniform random values shown in Figures 3.10(c) and 3.10(d). When using the filter with the relatively larger passband, we observe that the perimeter of the PSD is not as filtered, as shown in the three-dimensional and top-down perspectives of these filtered bivariate uniform random values in Figures 3.10(e) and 3.10(f). Consequently, the primary take-away point from this example is that the filtering operations of LTI systems can have a significant impact on the PSDs of random processes.

3.7 Narrowband Noise

Now that we have a solid foundation with respect to random variables and random processes, it is time to apply this knowledge to the application of narrowband transmissions for wireless communication systems. In general, most transmissions are designed to be *bandlimited* since there are constraints on the amount of wireless spectrum that any one transmission can use. These constraints are necessary since there is a limited amount of wireless spectrum available and a growing number of wireless applications and users seeking to use this spectrum for their transmissions.

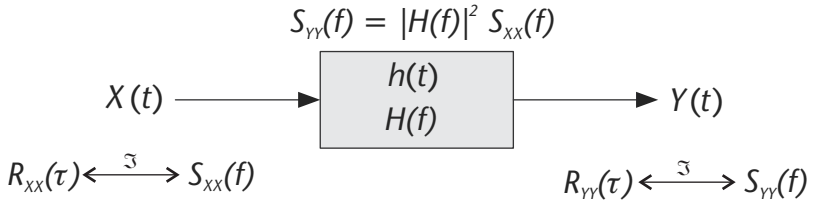

Figure 3.9 An example of how the an LTI system $h(t)$ can transform the PSD between the WSS random process input $X(t)$ and the WSS random process output $Y(t)$.

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Code 3.5 Information Source: **random_example.m**

```
% Define simulation parameters
L = 1000000; % Length of data streams
res_hist = 100; % Histogram resolution
cutoff_freq1 = 0.2; % Small passband LPF
cutoff_freq2 = 0.7; % large passband LPF
filt_coeffs1 = firls(13,[0 cutoff_freq1 cutoff_freq1+0.02 1],
                                                    [1 1 0 0]);
filt_coeffs2 = firls(13,[0 cutoff_freq2 cutoff_freq2+0.02 1],
                                                    [1 1 0 0]);

% Create input 2D Gaussian random variable
x_in = rand(1,L);
y_in = rand(1,L);

% Filter input random data stream
filt_output1 = filter(filt_coeffs1,1,(x_in+1j.*y_in));
x_out1 = real(filt_output1);
y_out1 = imag(filt_output1);
filt_output2 = filter(filt_coeffs2,1,(x_in+1j.*y_in));
x_out2 = real(filt_output2);
y_out2 = imag(filt_output2);
```

One of the key elements of a narrowband communication system is the narrowband filters at both the transmitter and receiver, which are designed to only allow only the modulated signals to pass. However, these narrowband filters also allow a portion of the noise intercepted at the receiver to pass through since it is very difficult to separate out the noise from the modulated signals. If it turns out that the noise is white (i.e., uncorrelated), then narrowband noise will take on the shaped of a cosine-modulated bandpass filter response. This is due to the fact that the white noise prior to filtering will have a PSD that is flat and spanning the entire frequency range from negative infinity to positive infinity. When processed by bandpass filters, the resulting narrowband noise PSD will take on the shape of the square of the magnitude response of the bandpass filters since the convolution of the noise with the filters in the time domain translates into the multiplication of the noise PSD with the filter magnitude response (see Section 3.6).

In terms of setting up a convenient mathematical framework to represent narrowband noise, there are two approaches: *in-phase/quadrature representation* and *envelope/phase representation*. Both approaches can describe a complex value x using the definition $x = Ae^{j\phi} = a + jb$, where $x \in \mathbb{C}$ and the envelope A, phase ϕ, inphase component a, and quadrature component b are real numbers $A, \phi, a, b \in \mathbb{R}$. The relationships between the in-phase/quadrature representation and the envelope/phase representation is described by the following:

$$A = \sqrt{a^2 + b^2}\phi = \tan^{-1}(b/a) \tag{3.56}$$

$$a = A\cos(\phi) \quad b = A\sin(\phi) \tag{3.57}$$

Thus, we can describe the in-phase/quadrature representation of narrowband noise in the complex baseband domain via the equation

$$\tilde{n}(t) = n_I(t) + jn_Q(t), \tag{3.58}$$

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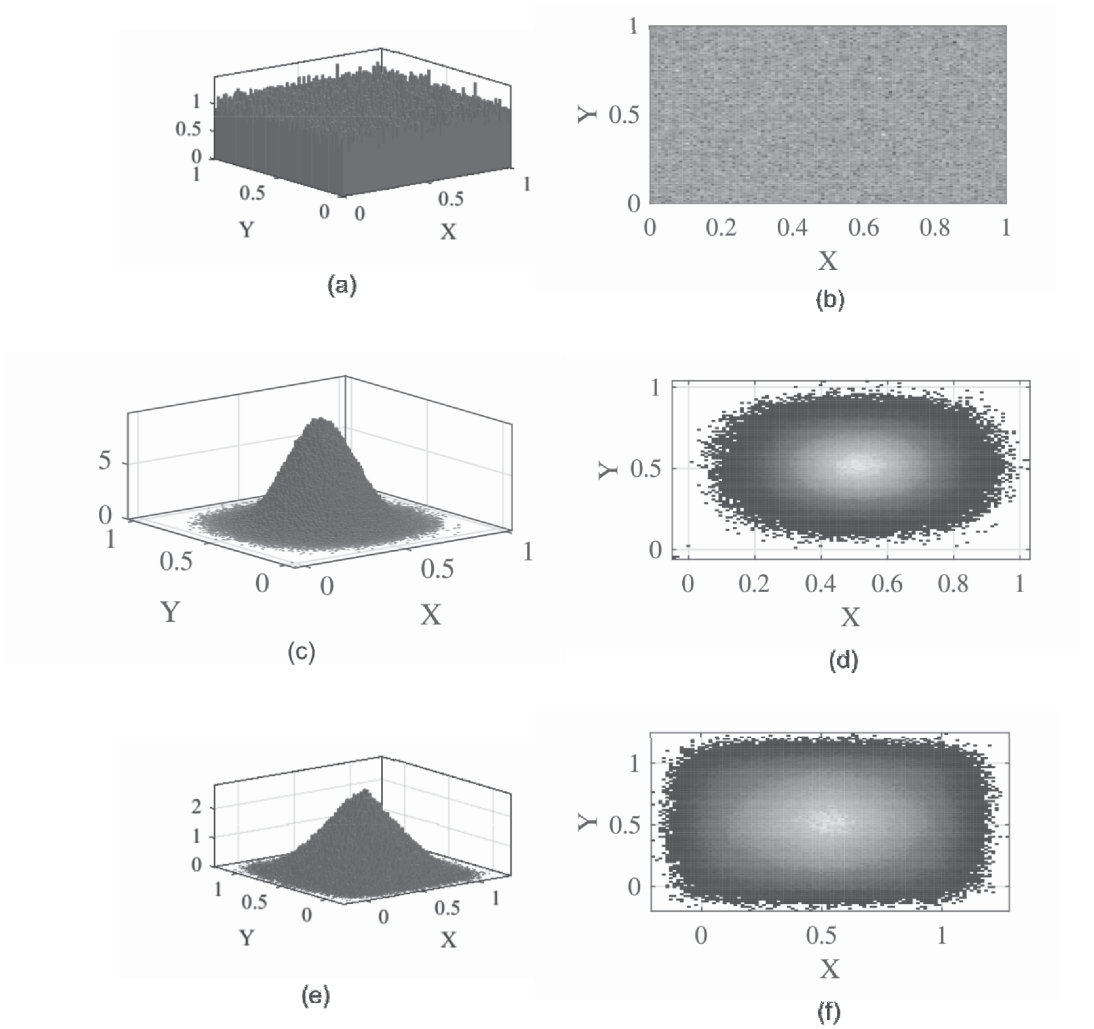

Figure 3.10 Example of how filtering can impact the output power spectral densities of random processes. (a) Power spectral density of input signal (3-D view), (b) power spectral density of input signal (top-down view), (c) power spectral density of output signal using filter coefficients 1 (3-D view), (d) power spectral density of output signal using filter coefficients 1 (top-down view), (e) power spectral density of output signal using filter coefficients 2 (3-D view), and (f) power spectral density of output signal using filter coefficients 2 (top-down view).

which can then be expressed in the bandpass version of a narrowband noise signal as

$$n(t) = \text{Real}\left\{\tilde{n}(t)e^{j2\pi f_c t}\right\}. \tag{3.59}$$

Using Euler's relationship; namely, $e^{j\omega} = \cos(\omega) + j\sin(\omega)$, we get the following expression:

$$n(t) = n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t). \tag{3.60}$$

Several important properties of the in-phase/quadrature representation are presented in Table 3.4.

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Table 3.4 Several Important Properties of In-phase/Quadrature Representation

Both $n_I(t)$ and $n_Q(t)$ have zero mean
If $n(t)$ is Gaussian, so are $n_I(t)$ and $n_Q(t)$
If $n(t)$ is stationary, $n_I(t)$ and $n_Q(t)$ are jointly stationary
PSD of $n_I(t)$ and $n_Q(t)$ equal to $S_{N_I}(f) = S_{N_Q}(f) = \begin{cases} S_N(f-f_c) + S_N(f+f_c), & -B \le f \le B \\ 0, & \text{otherwise} \end{cases}$
Both $n_I(t)$ and $n_Q(t)$ have the same variance as $n(t)$
If $n(t)$ is Gaussian and its PSD symmetric, then $n_I(t)$ and $n_Q(t)$ are statistically independent
The cross-spectral density between $n_I(t)$ and $n_Q(t)$ is purely imaginary, and for $-B \le f \le B$ it is equal to (zero otherwise) $S_{N_I N_Q}(f) = -S_{N_Q N_I}(f) = j(S_N(f+f_c) - S_N(f-f_c))$

Similarly, the complex baseband version of the envelope/phase representation of narrowband noise can be written as

$$n(t) = r(t)\cos(2\pi f_c t + \phi(t)) \tag{3.61}$$

where $r(t) = \sqrt{n_I(t) + n_Q(t)}$ is the envelope and $\phi(t) = \tan^{-1}(n_Q(t)/n_I(t))$ is the phase.

In terms of the relationship between the in-phase/quadrature representation and the envelope/phase representation with respect to their joint distributions, the results are very exciting. Suppose we define the joint PDF for $n_I(t)$ and $n_Q(t)$ as a bivariate Gaussian distribution equal to

$$f_{N_I N_Q}(n_I, n_Q) = \frac{1}{2\pi\sigma^2} e^{-\frac{n_I^2 + n_Q^2}{2\sigma^2}}. \tag{3.62}$$

It turns out that by using the relationships $n_I = r\cos(\phi)$ and $n_Q = r\sin(\phi)$ as well as a *Jacobian*, we obtain the following distributions:

$$f_{R\Phi}(r,\phi) = \begin{cases} \frac{r}{2\pi\sigma^2} e^{-\frac{r^2}{2\sigma^2}}, & r \ge 0 \text{ and } 0 \le \phi \le 2\pi \\ 0, & \text{otherwise} \end{cases} \tag{3.63}$$

which are equivalent to the combination of Rayleigh and uniform PDFs.

3.8 Application of Random Variables: Indoor Channel Model

An excellent example of where random variables are used to model a stochastic phenomenon in the design and implementation of a communication system is the indoor channel model proposed by Saleh and Valenzuela [3]. The premise of this channel model is that the indoor environment generates clusters of multipath components that result from the wireless signals reflecting off of the surrounding environment.

Mathematically speaking, they described these reflections of the transmitted signal intercepted at the receiver by the following expression:

$$h(t) = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \beta_{kl} e^{j\theta_{kl}} \delta(t - T_l - \tau_{kl}) \tag{3.64}$$

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where β_{kl} is the amplitude level of the kth ray of the lth multipath cluster, θ_{kl} is the phase value of the kth ray of the lth multipath cluster, T_l is the delay of the start of the lth multipath cluster, and τ_{kl} is the delay of the kth ray within the lth multipath cluster. In their work, Saleh and Valenzuela used channel measurements in order to characterize β_{kl} as a Rayleigh random variable, θ_{kl} as a uniform random variable, and both T_l and τ_{kl} as Poisson arrival processes with arrival rates Λ and λ. Graphically, this model can be illustrated using Figure 3.11.

3.9 Chapter Summary

In this chapter, a brief introduction to some of the key mathematical tools for analyzing and modeling random variables and random processes for communication systems has been presented. Of particular importance, the reader should understand how to mathematically manipulate Gaussian random variables, Gaussian random processes, and bivariate normal distributions since they frequently occur in digital communications and wireless data transmission applications. Furthermore, understanding how stationarity works and how to apply the EWK relations to situations involving random processes being filtered by LTI systems is vitally important, especially when dealing with the processing and treatment of received wireless signals by the communication system receiver.

3.10 Additional Readings

Although this chapter attempts to provide the reader with an introduction to some of the key mathematical tools needed to analyze and model random variables and random processes that frequently occur in many digital communication systems, the treatment of these mathematical tools is by no means rigorous or thorough. Consequently, the interested reader is encouraged to consult some of the available books that address this broad area in greater detail. For instance, the gold standard for any textbook on the subject of probability and random processes is by Papoulis and Pillai [1]. On the other hand, those individuals seeking to understand probability and random processes theory within the context of communication networks would potentially find the book by Leon-Garcia to be highly relevant [4].

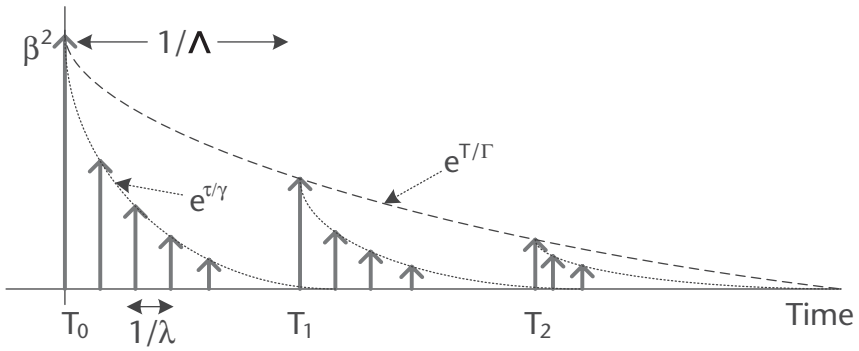

Figure 3.11 Illustration of the Saleh and Valenzuela statistical indoor channel model.

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For individuals who are interested in activities involving physical layer digital communication systems and digital signal processing, such as Wiener filtering, the book by Gubner would be a decent option given that many of the examples and problems included in this publication are related to many of the classic problems in communication systems [5]. Regarding books that possess numerous solved examples and explanations, those by Hsu [6] and Krishnan [7] would serve as suitable reading material. Finally, for those individuals who are interested in studying advanced topics and treatments of random processes, the book by Grimmett and Stirzaker would be a suitable publication [8].

References

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